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Moduli dependent μ-terms in a heterotic Standard Model

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Abstract: In this paper, we present a formalism for computing the non-vanishing Higgs μ-terms in a heterotic standard model. This is accomplished by calculating the cubic product of the cohomology groups associated with the vector bundle moduli (ϕ), Higgs (H) and Higgs conjugate ($\bar{H}$) superfields. This leads to terms proportional to $\phi H\bar{H}$ in the low energy superpotential which, for non-zero moduli expectation values, generate moduli dependent μ-terms of the form $\langle\phi\rangle H\bar{H}$. It is found that these interactions are subject to two very restrictive selection rules, each arising from a Leray spectral sequence, which greatly reduce the number of moduli that can couple to Higgs-Higgs conjugate fields. We apply our formalism to a specific heterotic standard model vacuum. The non-vanishing cubic interactions $\phi H\bar{H}$ are explicitly computed in this context and shown to contain only four of the nineteen vector bundle moduli.

Keywords: Superstring Vacua, Supersymmetric Standard Model, Higgs Physics, Superstrings and Heterotic Strings.

Contents

1. Introduction

Obtaining non-vanishing Higgs μ-terms and setting the scale of these interactions is one of the most important issues in realistic superstring model building [1]. In this paper, we present a formalism for computing these terms and explicitly demonstrate, within an important class of $E_8 \times E_8$ superstring vacua, that non-vanishing Higgs μ-terms are generated in the low energy effective theory. The scale of these μ-terms is set by the vacuum expectation values of a selected subset of vector bundle moduli.

In a series of papers [2 − 4], we presented a class of "heterotic standard model" vacua within the context of the $E_8 \times E_8$ heterotic superstring. The observable sector of a heterotic standard model vacuum is $N = 1$ supersymmetric and consists of a stable, holomorphic vector bundle, V, with structure group SU(4) over an elliptically fibered Calabi-Yau threefold, X, with a $\mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group[1]. Each such bundle admits a gauge connection

[1] In [2 − 4] we gave non-trivial checks on the slope-stability of the vector bundle V. Recently, stability of this bundle was rigorously proven in [5].

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which, in conjunction with a Wilson line, spontaneously breaks the observable sector E_8 gauge group down to the $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ Standard Model group times an additional gauged $\mathrm{U}(1)_{B-L}$ symmetry. The spectrum arises as the cohomology of the vector bundle V and is found to be precisely that of the minimal supersymmetric standard model (MSSM), with the exception of one additional pair of Higgs-Higgs conjugate superfields. These vacua contain *no exotic multiplets* and exist for both weak and strong string coupling. All previous attempts to find realistic particle physics vacua in superstring theories [6 − 19, 21, 20, 22] have run into difficulties. These include predicting extra vector-like pairs of light fields, multiplets with exotic quantum numbers in the low energy spectrum, enhanced gauge symmetries and so on. A heterotic standard model avoids all of these problems.

Elliptically fibered Calabi-Yau threefolds with $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group were first constructed in [23 − 25] and [26, 27] respectively. More recently, the existence of elliptic Calabi-Yau threefolds with $\mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group was demonstrated and their classification given in [28]. In [29 − 32], methods for building stable, holomorphic vector bundles with arbitrary structure group in E_8 over simply-connected elliptic Calabi-Yau threefolds were introduced. These results were greatly expanded in a number of papers [23 − 25, 33 − 35] and then generalized to elliptically fibered Calabi-Yau threefolds with non-trivial fundamental group in [25, 36, 26, 27]. To obtain a realistic spectrum, it was found necessary to introduce a new method [23 − 27] for constructing vector bundles. This method, which consists of building the requisite bundles by "extension" from simpler, lower rank bundles, was used for manifolds with $\mathbb{Z}_2$ fundamental group in [39, 40, 25, 37, 38] and in the heterotic standard model context in [28]. In [2 − 4, 37, 38], it was shown that to compute the complete low-energy spectrum of such vacua one must 1) evaluate the relevant sheaf cohomologies, 2) find the action of the finite fundamental group on these spaces and, finally, 3) tensor this with the action of the Wilson line on the associated representation. The low energy spectrum is the invariant cohomology subspaces under the resulting group action. This method was applied in [2 − 4] to compute the exact spectrum of all multiplets transforming non-trivially under the action of the low energy gauge group. The accompanying natural method of "doublet-triplet" splitting was also discussed. In a recent paper [41], a formalism was presented that allows one to enumerate and describe the multiplets transforming trivially under the low energy gauge group, namely, the vector bundle moduli.

Using the above work, one can construct a heterotic standard model and compute its entire low-energy spectrum. As mentioned previously, the observable sector spectrum is very realistic, consisting exclusively of the three chiral families of quarks/leptons (each family with a right-handed neutrino), two pairs of Higgs-Higgs conjugate fields and a small number of uncharged geometric and vector bundle moduli. However, finding a realistic spectrum is far from the end of the story. To demonstrate that the particle physics in these vacua is realistic, one must construct the exact interactions of these fields in the effective low energy Lagrangian. These interactions occur as two distinct types. Recall that the matter part of an $N = 1$ supersymmetric Lagrangian is completely described in terms of two functions, the superpotential and the Kahler potential. Of these, the superpotential, being a “holomorphic” function of chiral superfields, is much more amenable to computation using methods of algebraic geometry. The terms of the superpotential itself break into several different types, such as Higgs μ-terms and Yukawa couplings. In this paper, we begin our study of holomorphic interactions by presenting a formalism for computing Higgs μ-terms. We apply this method to calculate the non-vanishing μ-terms in a heterotic standard model.

Specifically, we do the following. In section 2, we review the relevant facts about the structure of heterotic standard model vacua and present the explicit example which we are going to use. The formalism for computing the low energy spectrum is briefly discussed and we give the results for our explicit choice of heterotic standard model vacuum. For example, the spectrum contains nineteen vector bundle moduli. Higgs μ-terms are then analyzed and shown to occur as the triple product involving two cohomology groups, one giving rise to vector bundle moduli (ϕ) and the other to Higgs (H) and Higgs conjugate ($\bar{H}$) fields in the effective low energy theory. For non-vanishing moduli expectation values, Higgs μ-terms of the form $\langle\phi\rangle H\bar{H}$ are generated in the superpotential. Section 3 is devoted to discussing the first Leray spectral sequence, which is associated with the projection of the covering threefold $\widetilde{X}$ onto the base space B_2. The Leray decomposition of a sheaf cohomology group into (p,q) subspaces is discussed and applied to the cohomology spaces relevant to Higgs μ-terms. It is shown that the triple product is subject to a (p,q) selection rule which severely restricts the allowed non-vanishing terms. Specifically, we find that only four out of the nineteen vector bundle moduli can participate in Higgs μ-terms. The second Leray decomposition, associated with the projection of the space B_2 onto its base $\mathbb{P}^1$, is presented in section 4. The decomposition of any cohomology space into its $[s,t]$ subspaces is discussed and applied to cohomologies relevant to Higgs μ-terms. We show that μ-terms are subject to yet another selection rule associated with the $[s,t]$ decomposition. Finally, it is demonstrated that the subspaces of cohomology that form non-vanishing cubic terms project non-trivially onto moduli, Higgs and Higgs conjugate fields under the action of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ group. This demonstrates that non-vanishing moduli dependent Higgs μ-terms proportional to $\langle\phi\rangle H\bar{H}$ appear in the low energy superpotential of a heterotic standard model.

Other holomorphic interactions in the superpotential, such as Yukawa couplings and moduli dependent“μ-terms” for possible exotic vector-like multiplets will be presented in up-coming publications. The more difficult issue of calculating the Kähler potentials in a heterotic standard model will be discussed elsewhere.

2. Preliminaries

2.1 Heterotic string on a Calabi-Yau manifold

The observable sector of an $E_8 \times E_8$ heterotic standard model vacuum consists of a stable, holomorphic vector bundle, V, with structure group SU(4) over a Calabi-Yau threefold, X, with fundamental group $\mathbb{Z}_3 \times \mathbb{Z}_3$. Additionally, the vacuum has a Wilson line, W, with $\mathbb{Z}_3 \times \mathbb{Z}_3$ holonomy. The SU(4) instanton breaks the low energy gauge group down to its

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commutant,

$$E_8 \xrightarrow{\mathrm{SU}(4)} \mathrm{Spin}(10)\ . \tag{2.1}$$

The Spin(10) group is then spontaneously broken by the holonomy group of W to

$$\mathrm{Spin}(10) \xrightarrow{\mathbb{Z}_3\times\mathbb{Z}_3} \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \times \mathrm{U}(1)_{B-L}\ . \tag{2.2}$$

In this way, the standard model gauge group emerges in the low energy effective theory multiplied by an additional U(1) gauge group whose charges correspond to $B-L$ quantum numbers.

The physical properties of this vacuum are most easily deduced not from X and V but, rather, from two closely related entities, which we denote by $\widetilde{X}$ and $\widetilde{V}$ respectively. $\widetilde{X}$ is a simply-connected Calabi-Yau threefold which admits a freely acting $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action such that

$$X = \widetilde{X} \Big/ \big(\mathbb{Z}_3 \times \mathbb{Z}_3\big). \tag{2.3}$$

That is, $\widetilde{X}$ is a covering space of X. Similarly, $\widetilde{V}$ is a stable, holomorphic vector bundle with structure group SU(4) over $\widetilde{X}$ which is equivariant under the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then,

$$V = \widetilde{V} \Big/ \big(\mathbb{Z}_3 \times \mathbb{Z}_3\big). \tag{2.4}$$

The covering space $\widetilde{X}$ for a heterotic standard model was discussed in detail in [28]. Here, it suffices to recall that $\widetilde{X}$ is a fiber product

$$\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2 \tag{2.5}$$

of two special $d\mathbb{P}_9$ surfaces B_1 and B_2. Thus, $\widetilde{X}$ is elliptically fibered over both surfaces with the projections

$$\pi_1 : \widetilde{X} \to B_1\,, \quad \pi_2 : \widetilde{X} \to B_2\,. \tag{2.6}$$

The surfaces B_1 and B_2 are themselves elliptically fibered over $\mathbb{P}^1$ with maps

$$\beta_1 : B_1 \to \mathbb{P}^1\,, \quad \beta_2 : B_2 \to \mathbb{P}^1\,. \tag{2.7}$$

Together, these projections yield the commutative diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B_1 \arrow[dr, "\beta_1"'] & & B_2 \arrow[dl, "\beta_2"] \\ & \mathbb{P}^1\ . & \end{tikzcd} \tag{2.8}$$

The invariant homology ring of each special $d\mathbb{P}_9$ surface is generated by two $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant curve classes f and t. Using the projections in eq. (2.6), these can be lifted to divisor classes

$$\tau_1 = \pi_1^{-1}(t_1)\,, \quad \tau_2 = \pi_2^{-1}(t_2)\,, \quad \phi = \pi_1^{-1}(f_1) = \pi_2^{-1}(f_2) \tag{2.9}$$

on $\widetilde{X}$. These three classes generate the invariant homology ring of $\widetilde{X}$.

2.2 The gauge bundle

The crucial ingredient in a heterotic standard model is the choice of the vector bundle $\widetilde{V}$. These bundles are constructed using a generalization of the method of *bundle extensions* [25, 27]. Specifically, $\widetilde{V}$ is the extension

$$0 \longrightarrow V_2 \longrightarrow \widetilde{V} \longrightarrow V_1 \longrightarrow 0 \tag{2.10}$$

of two rank two bundles V_1 and V_2 on $\widetilde{X}$. A solution for V_1 and V_2 is as follows. Define

$$\begin{aligned} V_1 &= \chi_2 \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2) \oplus \chi_2 \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2) \\ V_2 &= \mathcal{O}_{\widetilde{X}}(\tau_1-\tau_2) \otimes \pi_2^* W_2, \end{aligned} \tag{2.11}$$

where W_2 is an equivariant bundle in the extension space of

$$0 \longrightarrow \mathcal{O}_{B_2}(-2f_2) \longrightarrow W_2 \longrightarrow \chi_2 \mathcal{O}_{B_2}(2f_2) \otimes I_9 \longrightarrow 0 \tag{2.12}$$

and for the ideal sheaf I_9 of 9 points we take a generic $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbit. Here, χ_2 is one of the two natural one-dimensional representations of $\mathbb{Z}_3 \times \mathbb{Z}_3$ defined by

$$\chi_1(g_1) = \omega\,, \quad \chi_1(g_2) = 1\,; \qquad \chi_2(g_1) = 1\,, \quad \chi_2(g_2) = \omega\,, \tag{2.13}$$

where $g_{1,2}$ are the generators of the two $\mathbb{Z}_3$ factors, $\chi_{1,2}$ are two group characters of $\mathbb{Z}_3 \times \mathbb{Z}_3$, and $\omega = e^{\frac{2\pi i}{3}}$ is a third root of unity. The observable sector bundle $\widetilde{V}$ is then an equivariant element of the space of extensions defined in eq. (2.10). The vector bundle $\widetilde{V}$ passes the usual non-trivial checks on slope-stability, but we have not given a mathematically rigorous proof.

Let R be any representation of Spin(10) and $\mathrm{U}(\widetilde{V})_R$ the associated tensor product bundle of $\widetilde{V}$. Then, each sheaf cohomology space $H^i(\widetilde{X}, \mathrm{U}(\widetilde{V})_R)$, $i = 0, 1, 2, 3$ carries a specific representation of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Similarly, the Wilson line W manifests itself as a $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action on each representation R of Spin(10). As discussed in detail in [4], the low-energy particle spectrum is given by

$$\begin{aligned} \ker\left(\not{D}_{\widetilde{V}}\right) = &\Big(H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\big) \otimes \mathbf{45}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \oplus \Big(H^1\big(\widetilde{X}, \mathrm{ad}(\widetilde{V})\big) \otimes \mathbf{1}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \oplus \\ &\oplus \Big(H^1\big(\widetilde{X}, \widetilde{V}\big) \otimes \mathbf{16}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \oplus \Big(H^1\big(\widetilde{X}, \widetilde{V}^\vee\big) \otimes \overline{\mathbf{16}}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \oplus \Big(H^1\big(\widetilde{X}, \wedge^2 \widetilde{V}\big) \otimes \mathbf{10}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}\,, \end{aligned} \tag{2.14}$$

where the superscript indicates the invariant subspace under the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The invariant cohomology space $(H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \otimes \mathbf{45})^{\mathbb{Z}_3\times\mathbb{Z}_3}$ corresponds to gauge superfields in the low-energy spectrum carrying the adjoint representation of $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \times \mathrm{U}(1)_{B-L}$. The matter cohomology spaces, $(H^1(\widetilde{X}, \widetilde{V}) \otimes \mathbf{16})^{\mathbb{Z}_3\times\mathbb{Z}_3}$, $(H^1(\widetilde{X}, \widetilde{V}^\vee) \otimes \overline{\mathbf{16}})^{\mathbb{Z}_3\times\mathbb{Z}_3}$ and $(H^1(\widetilde{X}, \wedge^2\widetilde{V}) \otimes \mathbf{10})^{\mathbb{Z}_3\times\mathbb{Z}_3}$ were all explicitly computed in [4], leading to three chiral families of quarks/leptons (each family with a right-handed neutrino [42]), no exotic superfields and two vector-like pairs of Higgs-Higgs conjugate superfields respectively. The remaining cohomology space, $(H^1(\widetilde{X}, \mathrm{ad}(\widetilde{V})) \otimes \mathbf{1})^{\mathbb{Z}_3\times\mathbb{Z}_3}$, was recently computed in [41] and corresponds to nineteen vector bundle moduli.

2.3 Cubic terms in the superpotential

In this paper, we will focus on computing Higgs-Higgs conjugate μ-terms. First, note that in a heterotic standard model Higgs fields arise from eq. (2.14) as zero modes of the Dirac operator. Hence, they cannot have a "bare" μ-term in the superpotential proportional to $H\bar{H}$ with a constant coefficient. However, group theory does allow H and $\bar{H}$ to have cubic interactions with the vector bundle moduli of the form $\phi H\bar{H}$. If the moduli develop a non-vanishing vacuum expectation value, then these cubic interactions generate μ-terms of the form $\langle\phi\rangle H\bar{H}$ in the superpotential. Hence, in a heterotic standard model we expect Higgs μ-terms that are linearly dependent on the vector bundle moduli. Classically, no higher dimensional coupling of moduli to H and $\bar{H}$ is allowed.

It follows from eq. (2.14) that the 4-dimensional Higgs and moduli fields correspond to certain $\bar{\partial}$-closed $(0,1)$-forms on $\widetilde{X}$ with values in the vector bundle $\wedge^2\widetilde{V}$ and $\mathrm{ad}(\widetilde{V})$ respectively. Denote these forms by Ψ_H, $\Psi_{\bar{H}}$, and Ψ_ϕ. They can be written as

$$\Psi_H = \psi^{(H)}_{\bar{\iota},[ab]}\,\mathrm{d}\bar{z}^{\bar{\iota}}, \qquad \Psi_{\bar{H}} = \psi^{(\bar{H}),[ab]}_{\bar{\iota}}\,\mathrm{d}\bar{z}^{\bar{\iota}}, \qquad \Psi_\phi = [\psi^{(\phi)}_{\bar{\iota}}]_a{}^b\,\mathrm{d}\bar{z}^{\bar{\iota}}, \tag{2.15}$$

where a, b are valued in the SU(4) bundle $\widetilde{V}$ and $\{z^\iota, \bar{z}^{\bar{\iota}}\}$ are coordinates on the Calabi-Yau threefold $\widetilde{X}$. Doing the dimensional reduction of the 10-dimensional Lagrangian yields cubic terms in the superpotential of the 4-dimensional effective action. It turns out, see [11], that the coefficient of the cubic coupling $\phi H\bar{H}$ is simply the unique way to obtain a number out of the forms Ψ_H, $\Psi_{\bar{H}}$, Ψ_ϕ. That is,

$$W = \cdots + \lambda\phi H\bar{H} \tag{2.16}$$

where

$$\begin{aligned} \lambda &= \int_{\widetilde{X}} \Omega \wedge \mathrm{Tr}\Big[\Psi_\phi \wedge \Psi_H \wedge \Psi_{\bar{H}}\Big] = \\ &= \int_{\widetilde{X}} \Omega \wedge \Big([\psi^{(\phi)}_{\bar{\iota}}]_a{}^b\,\psi^{(H)}_{\bar{\kappa},[bc]}\,\psi^{(\bar{H}),[ca]}_{\bar{\lambda}}\Big)\mathrm{d}\bar{z}^{\bar{\iota}} \wedge \mathrm{d}\bar{z}^{\bar{\kappa}} \wedge \mathrm{d}\bar{z}^{\bar{\lambda}} \end{aligned} \tag{2.17}$$

and Ω is the holomorphic $(3,0)$-form. Mathematically, we are using the wedge product together with a contraction of the vector bundle indices to obtain a product

$$\begin{aligned} &H^1\Big(\widetilde{X}, \mathrm{ad}(\widetilde{V})\Big) \otimes H^1\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big) \otimes H^1\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big) \longrightarrow \\ &\qquad \longrightarrow H^3\Big(\widetilde{X}, \mathrm{ad}(\widetilde{V}) \otimes \wedge^2\widetilde{V} \otimes \wedge^2\widetilde{V}\Big) \longrightarrow H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) \end{aligned} \tag{2.18}$$

plus the fact that on the Calabi-Yau manifold $\widetilde{X}$

$$H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) = H^3\Big(\widetilde{X}, K_{\widetilde{X}}\Big) = H^{3,3}_{\bar{\partial}}\Big(\widetilde{X}\Big) = H^6\Big(\widetilde{X}\Big) \tag{2.19}$$

can be integrated over. If one were to use the heterotic string with the "standard embedding", then the above product would simplify further to the intersection of certain cycles in the Calabi-Yau threefold. However, in our case there is no such description.

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Hence, to compute μ-terms, we must first analyze the cohomology groups

$$H^1\Big(\widetilde{X}, \mathrm{ad}(\widetilde{V})\Big),\ H^1\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big),\ H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) \tag{2.20}$$

and the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on these spaces. We then have to evaluate the product in eq. (2.18). As we will see in the following sections, the two independent elliptic fibrations of $\widetilde{X}$ will force most, but not all, products to vanish.

3. The first elliptic fibration

As discussed in detail in [4], the cohomology spaces on $\widetilde{X}$ are obtained by using two Leray spectral sequences. In this section, we consider the first of these sequences corresponding to the projection

$$\widetilde{X} \xrightarrow{\pi_2} B_2. \tag{3.1}$$

For any sheaf $\mathcal{F}$ on $\widetilde{X}$, the Leray spectral sequence tells us that[2]

$$H^i\Big(\widetilde{X}, \mathcal{F}\Big) = \bigoplus_{p,q}^{p+q=i} H^p\Big(B_2, R^q\pi_{2*}\mathcal{F}\Big), \tag{3.2}$$

where the only non-vanishing entries are for $p = 0, 1, 2$ (since $\dim_{\mathbb{C}}(B_2) = 2$) and $q = 0, 1$ (since the fiber of $\widetilde{X}$ is an elliptic curve, therefore of complex dimension one). Note that the cohomologies $H^p(B_2, R^q\pi_{2*}\mathcal{F})$ fill out the 2×3 tableau[3]

$$\begin{array}{r|c|c|c|}
\cline{2-4}
{\scriptstyle q=1} & H^0\big(B_2, R^1\pi_{2*}\mathcal{F}\big) & H^1\big(B_2, R^1\pi_{2*}\mathcal{F}\big) & H^2\big(B_2, R^1\pi_{2*}\mathcal{F}\big) \\
\cline{2-4}
{\scriptstyle q=0} & H^0\big(B_2, \pi_{2*}\mathcal{F}\big) & H^1\big(B_2, \pi_{2*}\mathcal{F}\big) & H^2\big(B_2, \pi_{2*}\mathcal{F}\big) \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle p=0} & \multicolumn{1}{c}{\scriptstyle p=1} & \multicolumn{1}{c}{\scriptstyle p=2}
\end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, \mathcal{F}\Big), \tag{3.3}$$

where "$\Rightarrow H^{p+q}\big(\widetilde{X}, \mathcal{F}\big)$" reminds us which cohomology group the tableau is computing. Such tableaux are very useful in keeping track of the elements of Leray spectral sequences. As is clear from eq. (3.2), the sum over the diagonals yields the desired cohomology of $\mathcal{F}$. In the following, it will be very helpful to define

$$H^p\Big(B_2, R^q\pi_{2*}\mathcal{F}\Big) \equiv \big(p, q\big|\mathcal{F}\big). \tag{3.4}$$

Using this abbreviation, the tableau eq. (3.3) simplifies to

$$\begin{array}{r|c|c|c|}
\cline{2-4}
{\scriptstyle q=1} & \big(0,1\big|\mathcal{F}\big) & \big(1,1\big|\mathcal{F}\big) & \big(2,1\big|\mathcal{F}\big) \\
\cline{2-4}
{\scriptstyle q=0} & \big(0,0\big|\mathcal{F}\big) & \big(1,0\big|\mathcal{F}\big) & \big(2,0\big|\mathcal{F}\big) \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle p=0} & \multicolumn{1}{c}{\scriptstyle p=1} & \multicolumn{1}{c}{\scriptstyle p=2}
\end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, \mathcal{F}\Big). \tag{3.5}$$

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[2] In all the spectral sequences we are considering in this paper, higher differentials vanish trivially. Hence, the E_2 and E_∞ tableaux are equal and we will not distinguish them in the following. Furthermore, there are no extension ambiguities for $\mathbb{C}$-vector spaces.

[3] Recall that the zero-th derived push-down is just the ordinary push-down, $R^0\pi_{2*} = \pi_{2*}$.

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3.1 The first Leray decomposition of the volume form

Let us first discuss the (p, q) Leray tableau for the sheaf $\mathcal{F} = \mathcal{O}_{\widetilde{X}}$, which is the last term in eq. (2.20). Since the sheaf is trivial, it immediately follows that

$$\begin{array}{r|c|c|c|l} q{=}1 & 0 & 0 & \mathbf{1} & \\ \hline q{=}0 & \mathbf{1} & 0 & 0 & \Rightarrow H^{p+q}\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big). \\ \hline & p{=}0 & p{=}1 & p{=}2 & \end{array} \tag{3.6}$$

From eqs. (3.2) and (3.6) we see that

$$H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) = \big(2, 1\big|\mathcal{O}_{\widetilde{X}}\big) = \mathbf{1}, \tag{3.7}$$

where the $\mathbf{1}$ indicates that $H^3(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is a one-dimensional space carrying the trivial action of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

3.2 The first Leray decomposition of Higgs fields

Now consider the (p, q) Leray tableau for the sheaf $\mathcal{F} = \wedge^2 \widetilde{V}$, which is the second term in eq. (2.20). This was explicitly computed in [41] and is given by

$$\begin{array}{r|c|c|c|l} q{=}1 & 0 & \rho_{14} & 0 & \\ \hline q{=}0 & 0 & \rho_{14} & 0 & \Rightarrow H^{p+q}\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big), \\ \hline & p{=}0 & p{=}1 & p{=}2 & \end{array} \tag{3.8}$$

where ρ_{14} is the fourteen-dimensional representation

$$\rho_{14} = \Big(1 \oplus \chi_1 \oplus \chi_2 \oplus \chi_1^2 \oplus \chi_2^2 \oplus \chi_1\chi_2^2 \oplus \chi_1^2\chi_2\Big)^{\oplus 2} \tag{3.9}$$

of $\mathbb{Z}_3 \times \mathbb{Z}_3$. In general, it follows from eq. (3.2) that $H^1(\widetilde{X}, \wedge^2\widetilde{V})$ is the sum of the two subspaces $\big(0, 1\big|\wedge^2\widetilde{V}\big) \oplus \big(1, 0\big|\wedge^2\widetilde{V}\big)$. However, we see from the Leray tableau eq. (3.8) that the $\big(0, 1\big|\wedge^2\widetilde{V}\big)$ space vanishes. Hence,

$$H^1\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big) = \big(1, 0\big|\wedge^2\widetilde{V}\big). \tag{3.10}$$

Furthermore, eq. (3.8) tells us that

$$\big(1, 0\big|\wedge^2\widetilde{V}\big) = \rho_{14}. \tag{3.11}$$

3.3 The first Leray decomposition of the moduli

The (tangent space to the) moduli space of the vector bundle $\widetilde{V}$ is $H^1(\widetilde{X}, \operatorname{ad}(\widetilde{V}))$, the first term in eq. (2.20). First, note that $\operatorname{ad}(\widetilde{V})$ is defined to be the traceless part of $\widetilde{V} \otimes \widetilde{V}^\vee$. But the trace is just the trivial line bundle, whose first cohomology group vanishes. Therefore

$$H^1\Big(\widetilde{X}, \operatorname{ad}(\widetilde{V})\Big) = H^1\Big(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\Big) - \underbrace{H^1\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big)}_{=0}. \tag{3.12}$$

Since the action of the Wilson line on the $\mathbf{1}$ representation of Spin(10) is trivial, one need only consider the $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant subspaces of these cohomologies. That is, in the decomposition of the index of the Dirac operator, eq. (2.14), the moduli fields are contained in

$$\Big(H^1\big(\widetilde{X}, \mathrm{ad}(\widetilde{V})\big) \otimes \mathbf{1}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = H^1\Big(\widetilde{X}, \mathrm{ad}(\widetilde{V})\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = H^1\Big(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}. \tag{3.13}$$

In a previous paper [41], we computed the total number of moduli, but not their (p,q) degrees. However, this can be calculated in a straightforward manner.

To compute $H^1(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3}$, recall from [41] that the short exact bundle sequence eq. (2.10) generates a complex of intertwined long exact sequences which can be schematically represented by

$$\begin{tikzcd}
H^*\Big(V_2 \otimes V_1^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] \arrow[d] & H^*\Big(\widetilde{V} \otimes V_1^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] \arrow[d] & H^*\Big(V_1 \otimes V_1^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[d] \\
H^*\Big(V_2 \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] \arrow[d] & H^*\Big(\widetilde{V} \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] \arrow[d] & H^*\Big(V_1 \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[d] \\
H^*\Big(V_2 \otimes V_2^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] & H^*\Big(\widetilde{V} \otimes V_2^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] & H^*\Big(V_1 \otimes V_2^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3},
\end{tikzcd} \tag{3.14}$$

where $*$ means the complete cohomology with $* = 0,1,2,3$ and we have suppressed the base manifold $\widetilde{X}$ for notational simplicity. The (p,q) Leray tableaux for the "corner" cohomologies, marked by the dashed boxes in eq. (3.14), were calculated in [41]. Actually, the whole cohomology groups were determined, not just their invariant part. Restricting to the $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant subspace, we obtain

$$\begin{array}{r|c|c|c|}
\cline{2-4} q{=}1 & \mathbf{0} & \mathbf{0} & \mathbf{4} \\ \cline{2-4}
q{=}0 & \mathbf{4} & \mathbf{0} & \mathbf{0} \\ \cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2}
\end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, V_1 \otimes V_1^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}, \tag{3.15a}$$

$$\begin{array}{r|c|c|c|}
\cline{2-4} q{=}1 & \mathbf{4} & \mathbf{16} & \mathbf{0} \\ \cline{2-4}
q{=}0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2}
\end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, V_1 \otimes V_2^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}, \tag{3.15b}$$

$$\begin{array}{r|c|c|c|}
\cline{2-4} q{=}1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \cline{2-4}
q{=}0 & \mathbf{0} & \mathbf{16} & \mathbf{4} \\ \cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2}
\end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, V_2 \otimes V_1^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}, \tag{3.15c}$$

$$\begin{array}{r|c|c|c|}
\cline{2-4} q{=}1 & \mathbf{0} & \mathbf{3} & \mathbf{1} \\ \cline{2-4}
q{=}0 & \mathbf{1} & \mathbf{3} & \mathbf{0} \\ \cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2}
\end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, V_2 \otimes V_2^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}, \tag{3.15d}$$

where, as above, the $\mathbf{3}$, $\mathbf{4}$, and $\mathbf{16}$ denote the rank 3, 4, and 16 trivial representation of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Furthermore, the H^0 and, by Serre duality, the H^3 entries in the (p,q) Leray tableaux for the remaining cohomology groups in eq. (3.14) were computed in [41], where

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it was found that

$$
\begin{array}{r|c|c|c|}
\hline
q{=}1 & ** & ** & \mathbf{4} \\
\hline
q{=}0 & \mathbf{0} & ** & ** \\
\hline
\multicolumn{1}{r}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2}
\end{array}
\Rightarrow H^{p+q}\Big(\widetilde{X}, \widetilde{V} \otimes V_1^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}, \tag{3.16a}
$$

$$
\begin{array}{r|c|c|c|}
\hline
q{=}1 & ** & ** & \mathbf{0} \\
\hline
q{=}0 & \mathbf{4} & ** & ** \\
\hline
\multicolumn{1}{r}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2}
\end{array}
\Rightarrow H^{p+q}\Big(\widetilde{X}, V_1 \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}, \tag{3.16b}
$$

$$
\begin{array}{r|c|c|c|}
\hline
q{=}1 & ** & ** & \mathbf{0} \\
\hline
q{=}0 & \mathbf{1} & ** & ** \\
\hline
\multicolumn{1}{r}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2}
\end{array}
\Rightarrow H^{p+q}\Big(\widetilde{X}, \widetilde{V} \otimes V_2^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}, \tag{3.16c}
$$

$$
\begin{array}{r|c|c|c|}
\hline
q{=}1 & ** & ** & \mathbf{1} \\
\hline
q{=}0 & \mathbf{0} & ** & ** \\
\hline
\multicolumn{1}{r}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2}
\end{array}
\Rightarrow H^{p+q}\Big(\widetilde{X}, V_2 \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}, \tag{3.16d}
$$

$$
\begin{array}{r|c|c|c|}
\hline
q{=}1 & ** & ** & \mathbf{1} \\
\hline
q{=}0 & \mathbf{1} & ** & ** \\
\hline
\multicolumn{1}{r}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2}
\end{array}
\Rightarrow H^{p+q}\Big(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}. \tag{3.16e}
$$

The cohomology spaces on B_2 which are thus far uncalculated are denoted by $**$.

Our goal is to compute the entries in the (p,q) Leray tableaux for $H^1(\widetilde{V} \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3}$ at the positions $(0,1)$ and $(1,0)$ in eq. (3.16e). This can be accomplished as follows. First consider the $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant part of the lower horizontal long exact sequence in eq. (3.14). Restricting ourselves to the entries contributing to H^1, the exact sequence reads

$$
\begin{aligned}
\cdots &\longrightarrow H^0\big(V_1 \otimes V_2^\vee\big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \longrightarrow \\
&\longrightarrow H^1\big(V_2 \otimes V_2^\vee\big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \longrightarrow H^1\big(\widetilde{V} \otimes V_2^\vee\big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \longrightarrow H^1\big(V_1 \otimes V_2^\vee\big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \xrightarrow{\delta_1^\vee} \\
&\longrightarrow H^2\big(V_2 \otimes V_2^\vee\big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \longrightarrow \cdots .
\end{aligned} \tag{3.17}
$$

In [41] it was proven that

$$
H^0\big(V_1 \otimes V_2^\vee\big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = 0, \qquad \delta_1^\vee = 0. \tag{3.18}
$$

Hence, both coboundary maps vanish and we obtain the short exact sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(V_2 \otimes V_2^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} & \longrightarrow & H^1(\widetilde{V} \otimes V_2^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} & \longrightarrow & H^1(V_1 \otimes V_2^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} & \longrightarrow & 0 \\
\Vert & & \Uparrow & & \Uparrow & & \Uparrow & & \Vert \\
0 & \longrightarrow & \begin{array}{|c|c|c|}\hline \mathbf{0} & & \\ \hline & \mathbf{3} & \\ \hline\end{array} & \xrightarrow{\phi_1} & \begin{array}{|c|c|c|}\hline ** & & \\ \hline & ** & \\ \hline\end{array} & \xrightarrow{\phi_2} & \begin{array}{|c|c|c|}\hline \mathbf{4} & & \\ \hline & \mathbf{0} & \\ \hline\end{array} & \longrightarrow & 0.
\end{array} \tag{3.19}
$$

Now, on general grounds the coboundary maps in a long exact sequence increase the cohomology degree, while the interior maps preserve the cohomology degree. In particular, the maps ϕ_1 and ϕ_2 in eq. (3.19) must preserve the (p,q) degrees. The $(0,1)$ and $(1,0)$ entries in the $H^*(\widetilde{V} \otimes V_2^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3}$ Leray tableau can now be evaluated using the following general formula. Consider an exact sequence of linear spaces

$$\ldots \longrightarrow \mathcal{U} \xrightarrow{m_1} \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow \mathcal{X} \xrightarrow{m_2} \mathcal{Y} \longrightarrow \ldots\,, \tag{3.20}$$

where m_1 and m_2 are coboundary maps. Then

$$\dim_{\mathbb{C}}(\mathcal{W}) = \dim_{\mathbb{C}}(\mathcal{V}) + \dim_{\mathbb{C}}(\mathcal{X}) - \operatorname{rank}(m_1) - \operatorname{rank}(m_2). \tag{3.21}$$

This formula applies to any linear spaces, such as entire cohomology groups or their individual (p,q) Leray subspaces. Using eq. (3.21) for the $(0,1)$ and $(1,0)$ Leray degrees separately in eq. (3.19), we obtain the desired entries in the Leray tableau

$$\begin{array}{r|c|c|c|} \cline{2-4} q{=}1 & \mathbf{4} & & \\ \cline{2-4} q{=}0 & & \mathbf{3} & \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2} \end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, \widetilde{V} \otimes V_2^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}. \tag{3.22}$$

Second, consider the upper horizontal long exact sequence in eq. (3.14). Restricting ourselves to the entries contributing to H^1, this is given by

$$\begin{tikzcd}[column sep=small]
H^0(V_1\otimes V_1^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r, "d_2"] & H^1(V_2\otimes V_1^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] & H^1(\widetilde{V}\otimes V_1^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] & H^1(V_1\otimes V_1^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] & \cdots \\
\begin{array}{|c|c|c|} \hline & & \\ \hline \mathbf{4} & & \\ \hline \end{array} \arrow[u, Rightarrow] \arrow[r, shift left, "d_2|_{(0,1)}"] \arrow[r, shift right, "d_2|_{(1,0)}"'] & \begin{array}{|c|c|c|} \hline \mathbf{0} & & \\ \hline & \mathbf{16} & \\ \hline \end{array} \arrow[u, Rightarrow] \arrow[r] & \begin{array}{|c|c|c|} \hline ** & & \\ \hline & ** & \\ \hline \end{array} \arrow[u, Rightarrow] \arrow[r] & \begin{array}{|c|c|c|} \hline \mathbf{0} & & \\ \hline & \mathbf{0} & \\ \hline \end{array} \arrow[u, Rightarrow] \arrow[r] & \cdots\,.
\end{tikzcd} \tag{3.23}$$

The coboundary map d_2 on the left was shown in [41] to have $\operatorname{rank}(d_2) = 4$. In the context of the (p,q) Leray tableaux, it decomposes as

$$\operatorname{rank}\Big(d_2|_{(0,1)} : \mathbf{4} \to \mathbf{0}\Big) = 0, \quad \operatorname{rank}\Big(d_2|_{(1,0)} : \mathbf{4} \to \mathbf{16}\Big) = 4. \tag{3.24}$$

Again using eq. (3.21) for the $(0,1)$ and $(1,0)$ Leray degrees separately in eq. (3.23), we obtain the desired entries in the Leray tableau

$$\begin{array}{r|c|c|c|} \cline{2-4} q{=}1 & \mathbf{0} & & \\ \cline{2-4} q{=}0 & & \mathbf{12} & \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{p{=}0} & \multicolumn{1}{c}{p{=}1} & \multicolumn{1}{c}{p{=}2} \end{array} \Rightarrow H^{p+q}\Big(\widetilde{X}, \widetilde{V} \otimes V_1^\vee\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}. \tag{3.25}$$

From the results in eqs. (3.22) and (3.25), we can finally compute the (p,q) Leray subspaces that determine $H^1(\widetilde{V} \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3}$ in eq. (3.16e) using the middle vertical exact sequence of eq. (3.14)

$$\begin{tikzcd}
\cdots \arrow[r, "d_3"] & H^1(\widetilde{V} \otimes V_1^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] & H^1(\widetilde{V} \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r] & H^1(\widetilde{V} \otimes V_2^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} \arrow[r, "\delta_2"] & \cdots \\
\cdots \arrow[r] & \begin{array}{|c|c|c|} \hline \mathbf{0} & & \\ \hline & \mathbf{12} & \\ \hline \end{array} \arrow[u, Rightarrow] \arrow[r] & \begin{array}{|c|c|c|} \hline ** & & \\ \hline & ** & \\ \hline \end{array} \arrow[u, Rightarrow] \arrow[r] & \begin{array}{|c|c|c|} \hline \mathbf{4} & & \\ \hline & \mathbf{3} & \\ \hline \end{array} \arrow[u, Rightarrow] \arrow[r] & \cdots\,.
\end{tikzcd} \tag{3.26}$$

In [41], we calculated both coboundary maps d_3 and δ_2. It was found that they both vanish, that is

$$d_3 = \delta_2 = 0. \tag{3.27}$$

Using these results and eq. (3.21) for each of the two H^1 Leray subspace sequences in eq. (3.26), we find that the H^1 entries in the Leray tableau for $H^*(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3 \times \mathbb{Z}_3}$ are

q=1	**4**		
q=0		**15**	
	p=0	p=1	p=2

$$\Rightarrow H^{p+q}\Big(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}. \tag{3.28}$$

Note that

$$h^1(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = 4 + 15 = 19, \tag{3.29}$$

which is consistent with the conclusion in [41] that there are a total of nineteen vector bundle moduli. Now, however, we have determined the (p, q) decomposition of $H^1(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3 \times \mathbb{Z}_3}$ into the subspaces

$$H^1\Big(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = \big(0,1\big|\widetilde{V} \otimes \widetilde{V}^\vee\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \oplus \big(1,0\big|\widetilde{V} \otimes \widetilde{V}^\vee\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}, \tag{3.30}$$

where

$$\big(0,1\big|\widetilde{V} \otimes \widetilde{V}^\vee\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = \mathbf{4}, \quad \big(1,0\big|\widetilde{V} \otimes \widetilde{V}^\vee\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = \mathbf{15} \tag{3.31}$$

respectively.

3.4 The (p,q) selection rule

Having computed the decompositions of $H^3(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$, $H^1(\widetilde{X}, \wedge^2\widetilde{V})$ and $H^1(\widetilde{X}, \mathrm{ad}(\widetilde{V}))^{\mathbb{Z}_3 \times \mathbb{Z}_3}$ into their (p, q) Leray subspaces, we can now analyze the (p, q) components of the triple product

$$H^1\Big(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \otimes H^1\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big) \otimes H^1\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big) \longrightarrow H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) \tag{3.32}$$

given in eq. (2.18). Inserting eqs. (3.10) and (3.30), we see that

$$\begin{aligned}
&H^1\Big(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \otimes H^1\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big) \otimes H^1\Big(\widetilde{X}, \wedge^2\widetilde{V}\Big) = \\
&\quad = \Big(\big(0,1\big|\widetilde{V} \otimes \widetilde{V}^\vee\big) \oplus \big(1,0\big|\widetilde{V} \otimes \widetilde{V}^\vee\big)\Big) \otimes \big(1,0\big|\wedge^2\widetilde{V}\big) \otimes \big(1,0\big|\wedge^2\widetilde{V}\big) = \\
&\quad = \underbrace{\Big(\big(0,1\big|\widetilde{V}\otimes\widetilde{V}^\vee\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \otimes \big(1,0\big|\wedge^2\widetilde{V}\big) \otimes \big(1,0\big|\wedge^2\widetilde{V}\big)\Big)}_{\text{total } (p,q) \text{ degree } = (2,1)} \oplus \underbrace{\Big(\big(1,0\big|\widetilde{V}\otimes\widetilde{V}^\vee\big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \otimes \big(1,0\big|\wedge^2\widetilde{V}\big) \otimes \big(1,0\big|\wedge^2\widetilde{V}\big)\Big)}_{\text{total } (p,q) \text{ degree } = (3,0)}.
\end{aligned} \tag{3.33}$$

Because of the (p, q) degree, only the first term can have a non-zero product in

$$H^3\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\Big) = \big(2,1\big|\mathcal{O}_{\widetilde{X}}\big), \tag{3.34}$$

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see eq. (3.7). It follows that out of the $H^1(\widetilde{V} \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} = \mathbf{19}$ vector bundle moduli, only

$$\big(0,1\big|\widetilde{V} \otimes \widetilde{V}^\vee\big)^{\mathbb{Z}_3\times\mathbb{Z}_3} = \mathbf{4} \tag{3.35}$$

will form non-vanishing Higgs-Higgs conjugate μ-terms. The remaining fifteen moduli in the $\big(1,0\big|\widetilde{V} \otimes \widetilde{V}^\vee\big)^{\mathbb{Z}_3\times\mathbb{Z}_3}$ component have the wrong (p,q) degree to couple to a Higgs-Higgs conjugate pair. We refer to this as the (p,q) *Leray degree selection rule*. We conclude that the only non-zero product in eq. (3.32) is of the form

$$\big(0,1\big|\widetilde{V} \otimes \widetilde{V}^\vee\big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \otimes \big(1,0\big|\wedge^2\widetilde{V}\big) \otimes \big(1,0\big|\wedge^2\widetilde{V}\big) \longrightarrow \big(2,1\big|\mathcal{O}_{\widetilde{X}}\big). \tag{3.36}$$

Roughly what happens is the following. The Leray spectral sequence decomposes differential forms into the number p of legs in the direction of the base and the number q of legs in the fiber direction. The holomorphic $(3,0)$-form Ω has two legs in the base and one leg in the fiber direction. According to eq. (3.10), both 1-forms Ψ_H, $\Psi_{\bar{H}}$ corresponding to Higgs and Higgs conjugate have their one leg in the base direction. Therefore, the wedge product in eq. (2.17) can only be non-zero if the modulus 1-form Ψ_ϕ has its leg in the fiber direction, which only 4 out of the 19 moduli satisfy.

We conclude that due to a selection rule for the (p,q) Leray degree, the Higgs μ-terms in the effective low energy theory can involve only four of the nineteen vector bundle moduli.

4. The second elliptic fibration

So far, we only made use of the fact that our Calabi-Yau manifold is an elliptic fibration over the base B_2. But the $d\mathbb{P}_9$ surface B_2 is itself elliptically fibered over a $\mathbb{P}^1$. Consequently, there is yet another selection rule coming from the second elliptic fibration.

Therefore, we now consider the second Leray spectral sequence corresponding to the projection

$$B_2 \xrightarrow{\beta_2} \mathbb{P}^1. \tag{4.1}$$

For any sheaf $\widetilde{\mathcal{F}}$ on B_2, the Leray sequence tells us that

$$H^p\Big(B_2, \widetilde{\mathcal{F}}\Big) = \bigoplus_{s,t}^{s+t=p} H^s\Big(\mathbb{P}^1, R^t\beta_{2*}\widetilde{\mathcal{F}}\Big), \tag{4.2}$$

where the only non-vanishing entries are for $s = 0,1$ (since $\dim_{\mathbb{C}} \mathbb{P}^1 = 1$) and $t = 0,1$ (since the fiber of B_2 is an elliptic curve). The cohomologies $H^s(\mathbb{P}^1, R^t\beta_{2*}\widetilde{\mathcal{F}})$ fill out the 2×2 Leray tableau

$$\begin{array}{c|c|c|}
\hline
t{=}1 & H^0(\mathbb{P}^1, R^1\beta_{2*}\widetilde{\mathcal{F}}) & H^1(\mathbb{P}^1, R^1\beta_{2*}\widetilde{\mathcal{F}}) \\
\hline
t{=}0 & H^0(\mathbb{P}^1, \beta_{2*}\widetilde{\mathcal{F}}) & H^1(\mathbb{P}^1, \beta_{2*}\widetilde{\mathcal{F}}) \\
\hline
\multicolumn{1}{c}{} & \multicolumn{1}{c}{s{=}0} & \multicolumn{1}{c}{s{=}1}
\end{array} \Rightarrow H^{s+t}\Big(B_2, \widetilde{\mathcal{F}}\Big). \tag{4.3}$$

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As is clear from eq. (4.2), the sum over the diagonals yields the desired cohomology of $\widetilde{\mathcal{F}}$. Note that to evaluate the product eq. (3.36), we need the $[s,t]$ Leray tableaux for

$$\widetilde{\mathcal{F}} = R^1\pi_{2*}\big(\widetilde{V}\otimes\widetilde{V}^\vee\big), \quad \pi_{2*}\big(\wedge^2\widetilde{V}\big), \quad R^1\pi_{2*}\big(\mathcal{O}_{\widetilde{X}}\big). \tag{4.4}$$

In the following, it will be useful to define

$$H^s\Big(\mathbb{P}^1, R^t\beta_{2*}\Big(R^q\pi_{2*}(\mathcal{F})\Big)\Big) \equiv \big[s,t\big|q,\mathcal{F}\big]. \tag{4.5}$$

One can think of $\big[s,t\big|q,\mathcal{F}\big]$ as the subspace of $H^*\big(\widetilde{X},\mathcal{F}\big)$ that can be written as forms with q legs in the π_2-fiber direction, t legs in the β_2-fiber direction, and s legs in the base $\mathbb{P}^1$ direction.

4.1 The second Leray decomposition of the volume form

Let us first discuss the $[s,t]$ Leray tableau for the sheaf $\widetilde{\mathcal{F}} = R^1\pi_{2*}\big(\mathcal{O}_{\widetilde{X}}\big)$. Since $R^1\pi_{2*}\big(\mathcal{O}_{\widetilde{X}}\big) = K_{B_2}$, the canonical line bundle of B_2, it follows immediately that

$$\begin{array}{r|c|c|} \cline{2-3} t{=}1 & 0 & \mathbf{1} \\ \cline{2-3} t{=}0 & 0 & 0 \\ \cline{2-3} \multicolumn{1}{r}{} & \multicolumn{1}{c}{s{=}0} & \multicolumn{1}{c}{s{=}1} \end{array} \Rightarrow H^{s+t}\Big(B_2, R^1\pi_{2*}\big(\mathcal{O}_{\widetilde{X}}\big)\Big). \tag{4.6}$$

In our notation, this means that

$$H^2\Big(B_2, R^1\pi_{2*}\big(\mathcal{O}_{\widetilde{X}}\big)\Big) = \big[1,1\big|1,\mathcal{O}_{\widetilde{X}}\big] \tag{4.7}$$

has pure $[s,t]=[1,1]$ degree. We see from eqs. (4.6) and (4.2) that

$$H^3\Big(\widetilde{X},\mathcal{O}_{\widetilde{X}}\Big) = \big(2,1\big|\mathcal{O}_{\widetilde{X}}\big) = \big[1,1\big|1,\mathcal{O}_{\widetilde{X}}\big] = \mathbf{1}. \tag{4.8}$$

4.2 The second Leray decomposition of Higgs fields

Now consider the $[s,t]$ Leray tableau for the sheaf $\widetilde{\mathcal{F}} = \pi_{2*}\big(\wedge^2\widetilde{V}\big)$. This was explicitly computed in [41] and is given by

$$\begin{array}{r|c|c|} \cline{2-3} t{=}1 & (1\oplus\chi_1\oplus\chi_1^2\oplus\chi_1^2\oplus\chi_2^2\oplus\chi_1\chi_2^2)^{\oplus 2} & 0 \\ \cline{2-3} t{=}0 & 0 & (\chi_1^2\chi_2)^{\oplus 2} \\ \cline{2-3} \multicolumn{1}{r}{} & \multicolumn{1}{c}{s{=}0} & \multicolumn{1}{c}{s{=}1} \end{array} \Rightarrow H^{s+t}\Big(B_2, \pi_{2*}\big(\wedge^2\widetilde{V}\big)\Big). \tag{4.9}$$

This means that the 14 copies of the $\mathbf{10}$ of Spin(10) given in eq. (3.11) split as

$$H^1\Big(\widetilde{X},\wedge^2\widetilde{V}\Big) = \big(1,0\big|\wedge^2\widetilde{V}\big) = \big[0,1\big|0,\wedge^2\widetilde{V}\big] \oplus \big[1,0\big|0,\wedge^2\widetilde{V}\big], \tag{4.10}$$

where

$$\begin{aligned} \big[0,1\big|0,\wedge^2\widetilde{V}\big] &= \big(1\oplus\chi_1\oplus\chi_1^2\oplus\chi_1^2\oplus\chi_2^2\oplus\chi_1\chi_2^2\big)^{\oplus 2} \\ \big[1,0\big|0,\wedge^2\widetilde{V}\big] &= \big(\chi_1^2\chi_2\big)^{\oplus 2}. \end{aligned} \tag{4.11}$$

Note that

$$\big[0,1\big|0,\wedge^2\widetilde{V}\big] \oplus \big[1,0\big|0,\wedge^2\widetilde{V}\big] = \rho_{14} \tag{4.12}$$

in eq. (3.9), as it must.

4.3 The second Leray decomposition of the moduli

Finally, let us consider the $[s,t]$ Leray tableau for the moduli. We have already seen that, due to the (p,q) selection rule, only

$$\big(0,1\big|\widetilde{V}\otimes\widetilde{V}^{\vee}\big)^{\mathbb{Z}_3\times\mathbb{Z}_3}=\mathbf{4}\quad\subset H^1\Big(\widetilde{X},\widetilde{V}\otimes\widetilde{V}^{\vee}\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3} \tag{4.13}$$

out of the 19 moduli can occur in the Higgs-Higgs conjugate μ-term. Therefore, we are only interested in the $[s,t]$ decomposition of this subspace, that is, the degree 0 cohomology of the sheaf $R^1\pi_{2*}\big(\widetilde{V}\otimes\widetilde{V}^{\vee}\big)$. The corresponding Leray tableau is given by

$$\begin{array}{r|c|c|} \cline{2-3} t{=}1 & \phantom{\mathbf{4}} & \phantom{\mathbf{4}} \\ \cline{2-3} t{=}0 & \mathbf{4} & \\ \cline{2-3} \multicolumn{1}{r}{} & \multicolumn{1}{c}{s{=}0} & \multicolumn{1}{c}{s{=}1} \end{array} \Rightarrow H^{s+t}\Big(B_2,R^1\pi_{2*}\big(\widetilde{V}\otimes\widetilde{V}^{\vee}\big)\Big)^{\mathbb{Z}_3\times\mathbb{Z}_3}, \tag{4.14}$$

where the empty boxes are of no interest for our purposes. It follows that the 4 moduli of interest have $[s,t]$ degree $[0,0]$,

$$\big(0,1\big|\widetilde{V}\otimes\widetilde{V}^{\vee}\big)^{\mathbb{Z}_3\times\mathbb{Z}_3}=\big[0,0\big|1,\widetilde{V}\otimes\widetilde{V}^{\vee}\big]^{\mathbb{Z}_3\times\mathbb{Z}_3}=\mathbf{4}. \tag{4.15}$$

4.4 The $[s,t]$ selection rule

Having computed the decompositions of the relevant cohomology spaces into their $[s,t]$ Leray subspaces, we can now calculate the triple product eq. (2.18). The (p,q) selection rule dictates that the only non-zero product is of the form eq. (3.36). Now split each term in this product into its $[s,t]$ subspaces, as given in eqs. (4.8), (4.11), and (4.15) respectively. The result is

$$\begin{split}\big[0,0\big|1,\widetilde{V}\otimes\widetilde{V}^{\vee}\big]^{\mathbb{Z}_3\times\mathbb{Z}_3}\otimes\Big(\big[0,1\big|0,\wedge^2\widetilde{V}\big]\oplus\big[1,0\big|0,\wedge^2\widetilde{V}\big]\Big)\otimes&\\ \otimes\Big(\big[0,1\big|0,\wedge^2\widetilde{V}\big]\oplus\big[1,0\big|0,\wedge^2\widetilde{V}\big]\Big)&\longrightarrow\big[1,1\big|1,\mathcal{O}_{\widetilde{X}}\big].\end{split} \tag{4.16}$$

Clearly, this triple product vanishes by degree unless we choose the $\big[0,1\big|0,\wedge^2\widetilde{V}\big]$ from one of the $\big(1,0\big|\wedge^2\widetilde{V}\big)$ subspaces and $\big[1,0\big|0,\wedge^2\widetilde{V}\big]$ from the other. In this case, eq. (4.16) becomes

$$\big[0,0\big|1,\widetilde{V}\otimes\widetilde{V}^{\vee}\big]^{\mathbb{Z}_3\times\mathbb{Z}_3}\otimes\big[1,0\big|0,\wedge^2\widetilde{V}\big]\otimes\big[0,1\big|0,\wedge^2\widetilde{V}\big]\longrightarrow\big[1,1\big|1,\mathcal{O}_{\widetilde{X}}\big], \tag{4.17}$$

which is consistent.

4.5 Wilson lines

Recall that we have, in addition to the SU(4) instanton, also a Wilson line[4] turned on. Its effect is to break the Spin(10) gauge group down to the desired $SU(3)_C\times SU(2)_L\times U(1)_Y\times U(1)_{B-L}$ gauge group. Each fundamental matter field in the **10** can be broken to a Higgs

[4] In fact, we switch on a separate Wilson line for both $\mathbb{Z}_3$ factors in $\pi_1(X)=\mathbb{Z}_3\times\mathbb{Z}_3$.

field, a color triplet, or projected out. In particular, we are going to choose the Wilson line W so that its $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on a $\mathbf{10}$ representation of Spin(10) is given by

$$\mathbf{10} = \Big(\chi_1\chi_2^2 H \oplus \chi_1\chi_2 C\Big) \oplus \Big(\chi_1^2\chi_2 \bar{H} \oplus \chi_1^2\chi_2^2 \bar{C}\Big), \tag{4.18}$$

where

$$H = \big(\mathbf{1}, \mathbf{2}, 3, 0\big), \quad C = \big(\mathbf{3}, \mathbf{1}, -2, -2\big) \tag{4.19}$$

are the Higgs and color triplet representations of $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$ respectively.[5] Tensoring this with the cohomology space $H^1\big(\widetilde{X}, \wedge^2\widetilde{V}\big)$, we find the invariant subspace under the combined $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on the cohomology space and the Wilson line to be

$$\Big[H^1\big(\widetilde{X}, \wedge^2\widetilde{V}\big) \otimes \mathbf{10}\Big]^{\mathbb{Z}_3\times\mathbb{Z}_3} = \operatorname{span}\big\{H_1, H_2, \bar{H}_1, \bar{H}_2\big\}. \tag{4.20}$$

Hence, we find precisely two copies of Higgs and two copies of Higgs conjugate fields survive the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient. As required for any realistic model, all color triplets are projected out.

The new information now are the (p, q) and $[s, t]$ degrees of the Higgs fields. Using the decomposition of $H^1\big(\widetilde{X}, \wedge^2\widetilde{V}\big)$, we find

$$\begin{aligned}\Big[H^1\big(\widetilde{X}, \wedge^2\widetilde{V}\big) \otimes \mathbf{10}\Big]^{\mathbb{Z}_3\times\mathbb{Z}_3} &= \Big[\big(1,0\big|\wedge^2\widetilde{V}\big) \otimes \mathbf{10}\Big]^{\mathbb{Z}_3\times\mathbb{Z}_3} = \\ &= \underbrace{\Big[\big[0,1\big|0,\wedge^2\widetilde{V}\big] \otimes \mathbf{10}\Big]^{\mathbb{Z}_3\times\mathbb{Z}_3}}_{=\operatorname{span}\{\bar{H}_1,\bar{H}_2\}} \oplus \underbrace{\Big[\big[1,0\big|0,\wedge^2\widetilde{V}\big] \otimes \mathbf{10}\Big]^{\mathbb{Z}_3\times\mathbb{Z}_3}}_{=\operatorname{span}\{H_1,H_2\}}.\end{aligned} \tag{4.21}$$

The resulting degrees under the two Leray spectral sequences of the Higgs and Higgs conjugate fields are listed in table 1.

Field	(p, q)	$[s, t]$
H_1, H_2	$(1, 0)$	$[1, 0]$
$\bar{H}_1, \bar{H}_2$	$(1, 0)$	$[0, 1]$

Table 1: Degrees of the Higgs fields.

5. Higgs μ-terms

To conclude, we analyzed cubic terms in the superpotential of the form

$$\lambda_{iab}\phi_i H_a \bar{H}_b, \tag{5.1}$$

where

[5]The attentive reader will note that the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action of the Wilson line presented here differs from that given in [41]. Be that as it may, the low energy spectra of the two different actions are identical. However, for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action presented in this paper, there are non-vanishing Higgs μ-terms whereas all μ-terms vanish identically using the Wilson line action given in [41].

- λ_{iab} is a coefficient determined by the integral eq. (2.17),
- $\phi_i,\ i = 1, \ldots, 19$ are the vector bundle moduli,
- $H_a,\ a = 1, 2$ are the two Higgs fields, and
- $\bar{H}_b,\ b = 1, 2$ are the two Higgs conjugate fields.

We found that they are subject to two independent selection rules coming from the two independent torus fibrations. The first selection rule is that the total (p, q) degree is $(2, 1)$. According to table 1, $H_a \bar{H}_b$ already has (p, q) degree $(2, 0)$. Hence the moduli field ϕ_i must have degree $(0, 1)$. In eq. (3.31) we found that only 4 moduli ϕ_i, $i = 1, \ldots, 4$, have the right (p, q) degree. In other words, the majority of the coefficients vanishes,

$$\lambda_{iab} = 0, \qquad i = 5, \ldots, 19. \tag{5.2}$$

In principle, the second selection rule imposes independent constraints. It states that the total $[s, t]$ degree has to be $[1, 1]$. We showed that the allowed cubic terms $\phi_i H_a \bar{H}_b$, $i = 1, \ldots, 4$, all have the correct degree $[1, 1]$. Therefore, the (p, q) and $[s, t]$ selection rule allow μ-terms involving 4 out of the 19 vector bundle moduli. Cubic terms involving Higgs-Higgs conjugate fields and any of the remaining 15 moduli are forbidden in the superpotential.

When the moduli develop non-zero vacuum expectation values these superpotential terms generate Higgs μ-terms of the form

$$\lambda_{iab} \langle \phi_i \rangle H_a \bar{H}_b, \qquad i = 1, \ldots, 4,\ a = 1, 2,\ b = 1, 2. \tag{5.3}$$

Moreover, the coefficient λ_{iab} has no interpretation as an intersection number, and therefore has no reason to be constant over the moduli space. In general, we expect it to depend on the moduli. Of course, to explicitly compute this function one needs the Kähler potential which determines the correct normalization for all fields.

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